Branching bisimilarity is an equivalence indeed!

Twan Basten

Department of Mathematics and Computing Science, Eindhoven University of Technology, P.O. Box 513, NL-5600 MB Eindhoven, The Netherlands

This note presents a detailed proof of a result in the theory of concurrency semantics that is already considered folklore, namely that branching bisimilarity is an equivalence relation. The "simple proof," which in the literature is always assumed to exist, is shown to be incorrect. The proof in this note is based on the notion of a *semi-branching* bisimulation taken from [10]. Branching bisimilarity can equivalently be defined in terms of semi-branching bisimulations; the results suggest that such a definition is more intuitive than the original definition of [9].

Key words: Formal semantics; concurrency theory; branching bisimilarity

1 Introduction

In [9], branching bisimilarity has been introduced as an equivalence relation on processes that preserves the branching structure of processes. It distinguishes slightly more processes than the well-known observation equivalence as introduced by Milner [11]. Since its introduction, branching-bisimulation equivalence has rapidly gained popularity in the scientific literature on concurrency semantics (see for example [1–4,7,8,5,6]). However, the fact that it is indeed an equivalence relation is almost never proven. The only more or less explicit proof appears in [7,10,12]. In the other literature, including the original paper by Van Glabbeek and Weijland, the proof is simply omitted or it is claimed that it is "straightforward" or "easy to see." This note shows that the "proof" of [7,10,12] is incorrect and that it is *not* straightforward to show that branching bisimilarity is an equivalence, at least not for the original definition of branching bisimilarity.

The claim in [7,10,12] is that the relation composition of two branching bisimulations is again a branching bisimulation, which can be used to prove transitivity of branching bisimilarity. However, this note provides a counterexample invalidating that claim. Fortunately, there are several other ways to prove that branching bisimilarity is an equivalence relation. In [10], Van Glabbeek and Weijland already provide all the ingredients for a very elegant proof. In the proof of the so-called *stuttering lemma*, which without proof also appears in [9], for technical reasons, a slight variation of a branching bisimulation called a *semi-branching* bisimulation is introduced. It can be shown that the composition of two semi-branching bisimulations is again a semi-branching bisimulation, and therefore that semi-branching bisimilarity is an equivalence relation. Using this result plus a result from [10], it follows immediately that branching bisimilarity is an equivalence relation which coincides with semi-branching bisimilarity. Van Glabbeek already mentions this fact in [8]. The results show that the notion of branching bisimilarity can be defined in terms of semi-branching bisimulations; they also suggest that such a definition is more intuitive than the original definition of [9]. In addition, a semi-branching bisimulation can be defined in such a way that the distinction between internal and observable actions is made in an auxiliary relation, which greatly facilitates reasoning and yields very concise proofs. This is an extra argument for the conclusion that the notion of a semi-branching bisimulation yields a very elegant and intuitive definition of branching bisimilarity.

2 Branching and semi-branching bisimulations

This section contains some basic definitions. It introduces the notions of a branching bisimulation and a semi-branching bisimulation and explains the difference.

Definition 1 (Process space) *A process space over some set of actions $\mathcal{A}ct$ is a pair $(\mathcal{P}, \rightarrow)$, where $\mathcal{P}$ is a set of processes and $\_ \xrightarrow{\_} \_ \subseteq \mathcal{P} \times \mathcal{A}ct \times \mathcal{P}$ a ternary transition relation.*

In the remainder, assume that $\mathcal{P}$ is a process space over the set of actions $\mathcal{A}ct$ equal to $\mathcal{A} \cup \{\tau\}$, for some set of action symbols $\mathcal{A}$. The special symbol τ denotes the internal or silent action.

The following auxiliary relation expresses that a process can evolve into another process by executing a sequence of zero or more τ actions.

Definition 2 *The relation $\_ \twoheadrightarrow \_ \subseteq \mathcal{P} \times \mathcal{P}$ is defined as the smallest relation satisfying, for any $p, p', p'' \in \mathcal{P}$,*

$$p \twoheadrightarrow p \ ,$$

$$p \twoheadrightarrow p' \wedge p' \xrightarrow{\tau} p'' \ \Rightarrow \ p \twoheadrightarrow p'' \ .$$

Definition 3 (Branching bisimilarity [9,10]**)** *A binary relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is called a* branching bisimulation *if and only if, for any $p, p', q, q' \in \mathcal{P}$ and*

$\alpha \in \mathcal{A}ct$, *the following* transfer properties *are satisfied.*

$$p\mathcal{R}q \wedge p \xrightarrow{\alpha} p' \Rightarrow ((\alpha = \tau \wedge p'\mathcal{R}q) \vee$$
$$(\exists\, q', q'' \in \mathcal{P} :\ q \twoheadrightarrow q'' \xrightarrow{\alpha} q' \wedge p\mathcal{R}q'' \wedge p'\mathcal{R}q'))\ ,$$
$$p\mathcal{R}q \wedge q \xrightarrow{\alpha} q' \Rightarrow ((\alpha = \tau \wedge p\mathcal{R}q') \vee$$
$$(\exists\, p', p'' \in \mathcal{P} :\ p \twoheadrightarrow p'' \xrightarrow{\alpha} p' \wedge p''\mathcal{R}q \wedge p'\mathcal{R}q'))\ .$$

Two processes p and q are called branching bisimilar*, denoted $p \leftrightarrows_b q$, if and only if there exists a branching bisimulation relating p and q.*

The essence of this definition is depicted in Figure 1. Note that in [9,10] a branching bisimulation is required to be a *symmetric* relation. This yields a more concise definition, because it suffices to require the transfer property for only one direction. In this note, the above definition is chosen, because the relation composition of two (symmetric) relations is, in general, not symmetric. As the main results of this note concern compositions of (semi-) branching bisimulations, the above definition proves to be more convenient. As explained in [10], it is easy to see that the two definitions are equivalent. Obviously, any symmetric branching bisimulation is a branching bisimulation as defined above. Furthermore, it follows from the definition that the inverse of a branching bisimulation is also a branching bisimulation. Since also the union of two branching bisimulations is again a branching bisimulation, it follows that the union of a branching bisimulation and its inverse is a symmetric branching bisimulation. Hence, both definitions give rise to the same notion of branching bisimilarity.

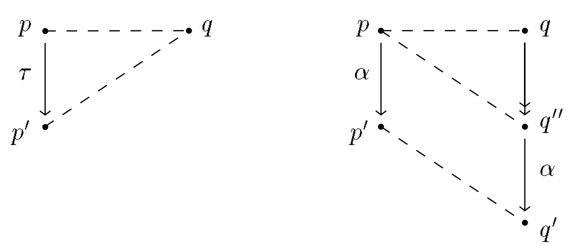

Fig. 1. A branching bisimulation.

Let, for any $p, p' \in \mathcal{P}$ and $\alpha \in \mathcal{A}ct$, $p \xrightarrow{(\alpha)} p'$ be an abbreviation of $p \xrightarrow{\alpha} p' \vee (\alpha = \tau \wedge p = p')$. That is, $p \xrightarrow{(\tau)} p'$ means zero or one τ steps and, for any $a \in \mathcal{A}$, $p \xrightarrow{(a)} p'$ is simply $p \xrightarrow{a} p'$.

Definition 4 (Semi-branching bisimilarity [10]**)** *A binary relation $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P}$ is called a* semi-branching bisimulation *if and only if, for any $p, p', q, q' \in \mathcal{P}$ and $\alpha \in \mathcal{A}ct$, the following transfer properties are satisfied.*

$$p\mathcal{R}q \wedge p \xrightarrow{\alpha} p' \Rightarrow (\exists\, q', q'' \in \mathcal{P} :\ q \twoheadrightarrow q'' \xrightarrow{(\alpha)} q' \wedge p\mathcal{R}q'' \wedge p'\mathcal{R}q')\ ,$$
$$p\mathcal{R}q \wedge q \xrightarrow{\alpha} q' \Rightarrow (\exists\, p', p'' \in \mathcal{P} :\ p \twoheadrightarrow p'' \xrightarrow{(\alpha)} p' \wedge p''\mathcal{R}q \wedge p'\mathcal{R}q')\ .$$

Two processes p and q are called semi-branching bisimilar*, denoted $p \leftrightarrows_{sb} q$,* *if and only if there exists a semi-branching bisimulation between p and q.*

Note that also the definition of a semi-branching bisimulation has a symmetric variant. Figure 2 [10] depicts the difference between a branching and a semi-branching bisimulation. Every branching bisimulation is a semi-branching bisimulation, whereas the converse is not true.

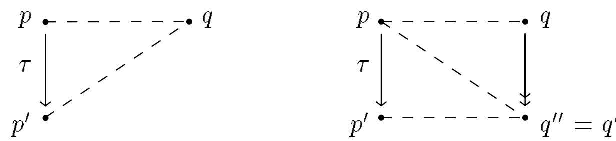

Fig. 2. A branching bisimulation (left) versus a semi-branching bisimulation (right).

3 Why semi-branching bisimulations?

In this section, it is shown that the relation composition of two branching bisimulations is not necessarily again a branching bisimulation. It is also shown that the relation composition of two semi-branching bisimulations is again a semi-branching bisimulation. Using this result, it is straightforward to prove that semi-branching bisimilarity is an equivalence relation. For arbitrary relations on processes $\mathcal{Q}$ and $\mathcal{R}$, the composition of $\mathcal{Q}$ and $\mathcal{R}$, denoted $\mathcal{Q} \circ \mathcal{R}$, is defined as $\{(p, r) \mid (\exists\, q : p\mathcal{Q}q \wedge q\mathcal{R}r)\}$.

Proposition 5 *The relation composition of two branching bisimulations is not necessarily again a branching bisimulation.*

Proof. Figure 3 shows two branching bisimulations $\mathcal{Q}$ and $\mathcal{R}$ relating the process that can execute three τ actions to itself. However, the composition of $\mathcal{Q}$ and $\mathcal{R}$, depicted on the right, is not a branching bisimulation. The action τ that causes the problem is depicted with a dotted arrow. Since none of the two pairs connected by a dotted line is an element of $\mathcal{Q} \circ \mathcal{R}$, $\mathcal{Q} \circ \mathcal{R}$ does not satisfy the requirements of Definition 3. $\square$

Note that the relation $\mathcal{Q} \circ \mathcal{R}$ is a *semi*-branching bisimulation. Also note that if a branching bisimulation is defined as a symmetric relation, then the above property is trivially correct, since the composition of two symmetric relations is in general not symmetric itself and, hence, cannot be a branching bisimulation. However, the counterexample shows that even the symmetric closure of the composition of $\mathcal{Q}$ and $\mathcal{R}$ is not always again a branching bisimulation.

The following lemma is needed for proving that the composition of two semi-branching bisimulations is a semi-branching bisimulation.

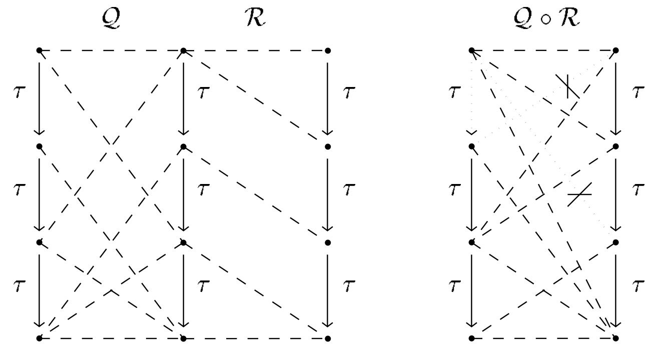

Fig. 3. $\mathcal{Q} \circ \mathcal{R}$ is not a branching bisimulation.

Lemma 6 *Let p and q be processes in $\mathcal{P}$ and let $\mathcal{R}$ be a semi-branching bisimulation such that $p\mathcal{R}q$. For any $p', q' \in \mathcal{P}$,*

$$p \twoheadrightarrow p' \Rightarrow (\exists\, q' \in \mathcal{P} :\ q \twoheadrightarrow q' \wedge p'\mathcal{R}q') \tag{1}$$
$$q \twoheadrightarrow q' \Rightarrow (\exists\, p' \in \mathcal{P} :\ p \twoheadrightarrow p' \wedge p'\mathcal{R}q') \tag{2}$$

Proof. Only Property (1) is proven. The proof of the other property is similar. The proof is by induction to the number of τ steps from p to p'.
Base case: Assume the number of τ steps from p to p' equals zero. Then, p and p' must be equal. Since $p\mathcal{R}q$ and $q \twoheadrightarrow q$, Property (1) is satisfied.
Inductive step: Assume $p \twoheadrightarrow p'$ in n τ steps, for some $n \geq 1$. It follows that there must exist a $p'' \in \mathcal{P}$ such that $p \twoheadrightarrow p''$ in $n-1$ τ steps and $p'' \xrightarrow{\tau} p'$. By the induction hypothesis, there exists a $q'' \in \mathcal{P}$ such that $q \twoheadrightarrow q''$ and $p''\mathcal{R}q''$. Since $p'' \xrightarrow{\tau} p'$ and $\mathcal{R}$ is a semi-branching bisimulation, there exist q' and q''' such that $q'' \twoheadrightarrow q''' \xrightarrow{(\tau)} q'$, $p''\mathcal{R}q'''$, and $p'\mathcal{R}q'$. Hence, $q \twoheadrightarrow q'$ and $p'\mathcal{R}q'$, which proves Property (1) also in this case. $\Box$

Proposition 7 *The relation composition of two semi-branching bisimulations is again a semi-branching bisimulation.*

Proof. Let $\mathcal{Q}$ and $\mathcal{R}$ be semi-branching bisimulations such that $p\mathcal{Q} \circ \mathcal{R}r$ for certain processes p and r in $\mathcal{P}$. Then, according to the definition of relation composition, there must be a process q in $\mathcal{P}$ such that $p\mathcal{Q}q$ and $q\mathcal{R}r$. First, assume there is a $p' \in \mathcal{P}$ such that $p \xrightarrow{\alpha} p'$, for some $\alpha \in \mathcal{A}ct$. It must be shown that

$$(\exists\, r', r'' \in \mathcal{P} :\ r \twoheadrightarrow r'' \xrightarrow{(\alpha)} r' \wedge p\mathcal{Q} \circ \mathcal{R}r'' \wedge p'\mathcal{Q} \circ \mathcal{R}r') \tag{3}$$

Since $p\mathcal{Q}q$, there exist $q', q'' \in \mathcal{P}$ such that $q \longrightarrow\!\!\!\!\rightarrow q'' \xrightarrow{(\alpha)} q'$, $p\mathcal{Q}q''$, and $p'\mathcal{Q}q'$. Since $q\mathcal{R}r$ and $q \longrightarrow\!\!\!\!\rightarrow q''$, Lemma 6 yields that there is an $r'' \in \mathcal{P}$ such that $r \longrightarrow\!\!\!\!\rightarrow r''$ and $q''\mathcal{R}r''$. Two cases can be distinguished.

(i) Assume $\alpha = \tau$ and $q'' = q'$. It follows immediately that $r \longrightarrow\!\!\!\!\rightarrow r'' \xrightarrow{(\alpha)} r''$, $p\mathcal{Q} \circ \mathcal{R}r''$, and $p'\mathcal{Q} \circ \mathcal{R}r''$. Hence, Property (3) is satisfied.

(ii) Assume $q'' \xrightarrow{\alpha} q'$. Since $q''\mathcal{R}r''$ and $\mathcal{R}$ is a semi-branching bisimulation, there must exist $r', r''' \in \mathcal{P}$ such that $r'' \longrightarrow\!\!\!\!\rightarrow r''' \xrightarrow{(\alpha)} r'$, $q''\mathcal{R}r'''$, and $q'\mathcal{R}r'$. It follows from $r \longrightarrow\!\!\!\!\rightarrow r''' \xrightarrow{(\alpha)} r'$, $p\mathcal{Q} \circ \mathcal{R}r'''$, and $p'\mathcal{Q} \circ \mathcal{R}r'$, that also in this case Property (3) is satisfied.

Second, for reasons of symmetry, it also follows that for any $r' \in \mathcal{P}$ and $\alpha \in \mathcal{A}ct$ such that $r \xrightarrow{\alpha} r'$, the transfer property is satisfied, which completes the proof. $\square$

Note that if a semi-branching bisimulation is defined as a symmetric relation on processes, then the above property does not hold. However, it is straightforward to show that the symmetric closure of the composition of two symmetric semi-branching bisimulations is again a symmetric semi-branching bisimulation.

Theorem 8 *Semi-branching bisimilarity, $\leftrightarrows_{sb}$, is an equivalence relation.*

Proof. It must be shown that semi-branching bisimilarity is reflexive, symmetric, and transitive.
Reflexivity: Let $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{P}$ be the identity relation on processes. Obviously, $\mathcal{I}$ is a semi-branching bisimulation such that, for any $p \in \mathcal{P}$, $p\mathcal{I}p$. Hence, $p \leftrightarrows_{sb} p$, which proves reflexivity of $\leftrightarrows_{sb}$.
Symmetry: Let $p, q \in \mathcal{P}$. Assume that $\mathcal{R}$ is a semi-branching bisimulation such that $p\mathcal{R}q$. Since the inverse of $\mathcal{R}$ is a semi-branching bisimulation relating q and p, semi-branching bisimilarity is symmetric.
Transitivity: Transitivity of $\leftrightarrows_{sb}$ follows immediately from Proposition 7. $\square$

4 Branching bisimilarity is an equivalence relation

There are several ways to show that, despite Proposition 5, branching bisimilarity is an equivalence relation. In [10], it is shown that two processes are branching bisimilar if and only if they have the same hypertraces. From this result, it immediately follows that branching bisimilarity is an equivalence relation. In [6], De Nicola en Vaandrager provide a modal characterization for branching bisimilarity. Again, it follows that branching bisimilarity is an equivalence. A third way is based on the fact that for any two branching bisimilar processes, there exists a *largest* branching bisimulation relating these processes. The relation composition of two largest branching bisimulations is again a branching bisimulation which again yields the desired result. All three proofs are based on one and the same lemma, namely the *stuttering lemma* given in [9,10]. The proof of this lemma in [10] yields even a fourth proof for the fact that branching bisimilarity is an equivalence. It shows that two processes are related by a branching bisimulation if and only if they are related by a semi-branching bisimulation. Since semi-branching bisimilarity is an equivalence, it follows immediately that branching bisimilarity must also be an equivalence. It also follows that the two equivalence relations coincide. This result is also mentioned in [8]. The remainder of this section presents this last proof in more detail.

Definition 9 (Stuttering property [9,10]**)** *Let* $\mathcal{R}$ *be a binary relation on* $\mathcal{P}$*.* $\mathcal{R}$ *is said to satisfy the* stuttering property *if and only if for all* $k \geq 0$ *and* $p, p_0, \ldots, p_k, q, q_0, \ldots, q_k \in \mathcal{P}$*:*
$p_0 \mathcal{R} q$*,* $p_k \mathcal{R} q$*, and* $p_0 \xrightarrow{\tau} p_1 \xrightarrow{\tau} \ldots \xrightarrow{\tau} p_k$ *implies that* $p_i \mathcal{R} q$ *for all* i*,* $1 \leq i < k$*;*
$p \mathcal{R} q_0$*,* $p \mathcal{R} q_k$*, and* $q_0 \xrightarrow{\tau} q_1 \xrightarrow{\tau} \ldots \xrightarrow{\tau} q_k$ *implies that* $p \mathcal{R} q_i$ *for all* i*,* $1 \leq i < k$*.*

A (semi-) branching bisimulation satisfies the stuttering property if and only if the following two properties are satisfied: for any processes p_0 and p_k such that p_0 can evolve into p_k by only executing τ actions, p_0 related to q and p_k related to q implies that all intermediate processes are also related to q; for any processes q_0 and q_k such that q_0 can evolve into q_k by only executing τ actions, p related to q_0 and p related to q_k implies that p is also related to all intermediate processes.

Corollary 10 *Any semi-branching bisimulation satisfying the stuttering property is a branching bisimulation.*

Recall that the union of arbitrarily many (semi-) branching bisimulations yields again a (semi-) branching bisimulation. Therefore, for any pair of (semi-) branching bisimilar processes p and q in $\mathcal{P}$, there exists a *largest* (semi-) branching bisimulation $\mathcal{R}$ such that $p \mathcal{R} q$. The following lemma is taken from [10].

Lemma 11 *For any two processes* p *and* q *in* $\mathcal{P}$ *such that* $p \leftrightarroweq_{sb} q$*, the largest semi-branching bisimulation between* p *and* q *satisfies the stuttering property.*

As a consequence, the largest semi-branching bisimulation between p and q is a branching bisimulation. Since any branching bisimulation is a semi-branching bisimulation, this yields the following two corollaries.

Corollary 12 *Two processes are related by a branching bisimulation if and only if they are related by a semi-branching bisimulation.*

Corollary 13 *Branching bisimilarity,* $\leftrightarrows_b$ *, is an equivalence relation, which coincides with semi-branching bisimilarity.*

5 Concluding remarks

This note presents a proof for the fact that branching bisimilarity is an equivalence relation. Moreover, it shows that for the *original* definition of branching bisimilarity, the proof is everything but straightforward, as is often suggested in the literature. The only explicit "proof" published so far is shown to be incorrect. Proposition 5, which is very counterintuitive, in combination with Proposition 7 and the fact that branching bisimilarity and semi-branching bisimilarity coincide suggest that the notion of branching bisimilarity can be defined in a more intuitive and elegant way if the notion of a semi-branching bisimulation is used instead of a branching bisimulation. The concise definition of a semi-branching bisimulation, where the distinction between internal and observable actions is made in an auxiliary relation supports this conclusion.

As a final remark, note that the results in this note also hold for several variations of branching bisimilarity, such as *rooted* branching bisimilarity, which is also introduced in [9], and (rooted) branching bisimilarity with termination predicates, which is often used in the area of process algebra (See for example [1]).

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